Off-forward parton distributions and impact parameter dependence of parton structure

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Abstract

The connection between parton distributions as a function of the impact parameter and off-forward parton distributions is discussed in the limit of vanishing skewedness parameter ζ, i.e. when the off-forwardness is purely transverse. For the 2nd moment it is also illustrated how to relate $\zeta \neq 0$ data to $\zeta = 0$ data, which is important for experimental measurements of these observables.

1 Introduction

Deeply virtual Compton scattering experiments in the Bjorken limit allow measuring generalized or off-forward parton distributions (OFPDs) [1,2]

$$f_\zeta(x,t) \equiv \int \frac{dx^-}{4\pi} \langle p'|\bar{\psi}(0)\gamma^+\psi(x^-)|p\rangle e^{ixp^+x^-}, \tag{1}$$

where $x^\pm = x^0 \pm x^3$ and $p^+ = p^0 + p^3$ refer to the usual light-cone components and $t \equiv q^2 = (p - p')^2$ is the invariant momentum transfer. The "off-forwardness" (or skewedness) in Eq. (1) is defined to be $\zeta \equiv \frac{q^+}{p^+}$. From the point of view of parton physics in the infinite momentum frame, these OFPDs have the physical meaning of the amplitude for the process that a quark is taken out of the nucleon with momentum fraction x and then it is inserted back into the nucleon with a four momentum transfer q^μ [3]. OFPDs play a dual roles and in a certain sense they interpolate between form factors and conventional parton distribution functions (PDFs) [1,2]: for $\zeta = t = 0$ one recovers conventional PDFs, i.e. longitudinal momentum distributions in the infinite momentum frame (IMF), while when one integrates $f_\zeta(x,t)$ over x, one obtains a form factor, i.e. the Fourier transform of a coordinate space density (in the Breit frame!). One of the new physics insights that one can learn from these OFPDs is the angular momentum distribution [1], but apart from that

people are just starting to explore what kind of new physics one can learn from studying these generalized parton distributions. The main reason, why the physical interpretation of OFPDs is still somewhat obscure is due to the fact that the initial and final state in Eq. (1) are not the same and therefore, in general, the OFPDs $f_\zeta(x, t)$ cannot be interpreted as a 'density' but rather their physical significance is that of a probability amplitude.

In this note, we will discuss the limit $\zeta \to 0$, but $t \neq 0$, where

$$f(x, t) \equiv f_{\zeta=0}(x, t) \tag{2}$$

does have a simple interpretation in terms of a density, namely as the Fourier transform of the light-cone momentum/impact parameter density w.r.t. the impact parameter $\vec{b}_\perp$.

2   Impact parameter dependent PDFs and OFPDs

PDFs are usually defined as matrix elements between plane wave states, which extend throughout space. Therefore, before we can introduce the notion of impact parameter dependent PDFs, we need to define localized wave packets. For our purposes, it is most suitable to consider a wave packet $|\Psi\rangle$ which is chosen such that it has a sharp longitudinal momentum p_z, but whose position is a localized wave packet in the transverse direction

$$|\Psi\rangle = \int \frac{d^2 p_\perp}{\sqrt{2E_{\vec{p}}(2\pi)^2}} \Psi(\vec{p}_\perp) |\vec{p}\rangle . \tag{3}$$

Although this definition is completely general, what we have in mind for the states $|p\rangle$ are for example nucleon states, which are of course extended objects, i.e. Eq. (3) describes wave packets of particles that are themselves already extended objects. We will come back to this point further below. Clearly,

$$F_\Psi(x, \vec{b}_\perp) \equiv \int \frac{dx^-}{4\pi} \langle\Psi|\bar{\psi}(0^-, \vec{b}_\perp)\gamma^+\psi(x^-, \vec{b}_\perp)|\Psi\rangle e^{ixp^+x^-}, \tag{4}$$

describes the probability to find partons with momentum fraction x at transverse (position) coordinate $\vec{b}_\perp$ in this wave packet. Note that in Eq. (4) we have implicitly assumed that we work in light-cone gauge $A^+ = 0$. In any other gauge, one needs to insert a (straight line) 'gauge string' connecting the points $(0^-, \vec{b}_\perp)$ and $(x^-, \vec{b}_\perp)$ in order to render Eq. (4) manifestly gauge invariant.

What we will show in the following is that, for a suitably localized wave packet, $F_\Psi(x, \vec{b}_\perp)$ can be related to OFPDs with $\zeta = 0$. Using Eq. (3), one finds

$$
\begin{aligned}
f_\Psi(x, \vec{q}_\perp) &\equiv \int d^2 q_\perp e^{-i\vec{q}_\perp \cdot \vec{b}_\perp} F_\Psi(x, \vec{b}_\perp) \\
&= \int \frac{d^2 p_\perp \Psi^*(\vec{p}_\perp')\Psi(\vec{p}_\perp)}{\sqrt{2E_{\vec{p}} 2E_{\vec{p}'}}} \int dx^- e^{ixp^+ x^-} \langle p'|\bar{\psi}(0^-, \vec{0}_\perp)\psi(x^-, \vec{0}_\perp)|p\rangle \\
&= \int \frac{d^2 p_\perp \Psi^*(\vec{p}_\perp')\Psi(\vec{p}_\perp)}{\sqrt{2E_{\vec{p}} 2E_{\vec{p}'}}} f_\zeta(x, q^2).
\end{aligned}
\tag{5}
$$

where $\vec{p}_\perp' = \vec{p}_\perp + \vec{q}_\perp$ and $p_z' = p_z$, i.e. $\zeta = 0$.

The physics of Eq. (5) is the following: the non-trivial intrinsic structure of the target particle is expressed in the OFPD $f_\zeta(x, q^2)$. However, as the convolution of $f_\zeta(x, q^2)$ with the wave function reflects, the impact parameter dependent PD in the state Ψ gets spread out in position space due to the fact that the wave function Ψ does in general have a nonzero width in position space.

Intuitively, one would like to chose a wave packet that is point-like in position space (i.e. constant in momentum space) so that the $\vec{b}_\perp$-dependence in Eq. (4) is only due to the intrinsic structure of the target particle but not due to the wave packet used to 'nail it down'. However, one need to be careful in this step (and being able to properly address these issues was the sole reason for working with wave packets) because as soon as one localizes a particle to a region of space smaller than its Compton wavelength, its motion within the wave packet becomes relativistic and therefore the structure of the particle gets affected by Lorentz contraction as well as other relativistic effects.

2.1 Nonrelativistic limit

It is very instructive to consider the nonrelativistic (NR) limit first, where none of these complications occur. Formally, the simplification arises since $E_{\vec{p}} = E_{\vec{p}'} = M$ and therefore $q^2 = -\vec{q}^2 = -\vec{q}_\perp^2$ First of all, this means that one can pull $f(x, -\vec{q}_\perp^2)$ out of the integral in Eq. (5), yielding

$$
f_\Psi(x, \vec{q}_\perp) = f(x, -\vec{q}_\perp^2) \int \frac{d^2 p_\perp \Psi^*(\vec{p}_\perp')\Psi(\vec{p}_\perp)}{2M}.
\tag{6}
$$

In order to proceed further, we choose a wave packet that is very localized in transverse position space. Specifically, we choose a packet whose width in transverse momentum space is much larger than a typical QCD scale. That way, the $\vec{q}_\perp$-dependence on the r.h.s. of Eq. (6) is mostly due to the matrix

3

element and not due to the wave packet Ψ. Therefore, by making the wave packet very localized in position space one obtains

$$f_\Psi(x, \vec{q}_\perp) = f(x, -\vec{q}_\perp^2). \tag{7}$$

The dependence on the detailed shape of the wave packet has disappeared once it is chosen localized enough and it is thus legitimate to identify the Fourier transformed (w.r.t. $\vec{q}_\perp$) $\zeta = 0$ OFPD with the impact parameter dependence of the parton distribution in the target particle itself.

2.2 Relativistic corrections

In order to have a unique definition of impact parameter dependent PDFs, i.e. a definition which does not depend on the exact shape of the wave packet, one would like to make the wave packet as small as possible in position space and certainly much smaller than the spatial extension of the hadron in the $\perp$ direction — otherwise Eq. (4) is dominated by the shape of the wave packet and not by the intrinsic $\vec{b}_\perp$-dependence. However, once the wave packet is smaller than about a Compton wavelength of the target then relativistic corrections can no longer be ignored and may even become more important as the corrections due to the spatial extension of the wave packet.

This problem is well known from form factors, where the identification of the form factor with the Fourier transform of a charge distribution in position space is uniquely possible only for momenta that are much smaller than the target mass. One can also rephrase this statement in position space by saying that the identification of the Fourier transform of the form factor with a charge distribution works only if one looks at scales larger than about one Compton wavelength of the target. Details below this scale may depend on the Lorentz frame and are therefore not physical.

For OFPDs the problem is very similar. If one works for example in the rest frame then one faces the same kind of corrections that also affect the form factors, in the sense that the $\perp$ resolution is limited to about one Compton wavelength. More details can be found in Ref. [4] and are omitted here since the natural frame to interpret parton distribution functions is the infinite momentum frame (IMF), and as we will discuss in the next section, these relativistic corrections play no important role in the IMF.

2.3   Infinite momentum frame

In the following we chose a wave packet as in Eq. (3) but with $p_z \gg M$, where M is the target mass. Then one can for example expand [1]

$$E_{\vec{p}+\vec{q}} \approx |p_z| + \frac{\vec{p}_\perp^2 + 2\vec{p}_\perp \vec{q}_\perp + \vec{q}_\perp^2}{2|p_z|} \tag{8}$$

and therefore the energy transfer

$$E_{\vec{p}+\vec{q}} - E_{\vec{p}} \sim \frac{2\vec{p}_\perp \vec{q}_\perp + \vec{q}_\perp^2}{2|p_z|} \tag{9}$$

vanishess as $p_z \to \infty$. A more detailed analysis works as follows: Let us denote the typical momentum scale in the wave packet by $\Lambda_{\vec{p}_\perp}$ and the $\perp$ resolution that we are interested in by $\Lambda_{\vec{q}_\perp}$ (normally, $\Lambda_{\vec{q}_\perp}$ will be a typical hadronic momentum scale, i.e. on the order of $1 GeV$). Then of course what we need to satisfy is $\Lambda_{\vec{p}_\perp} \gg \Lambda_{\vec{q}_\perp}$. At the same time we need $|p_z| \gg \Lambda_{\vec{p}_\perp}, \Lambda_{\vec{q}_\perp}$, but this is no problem if we go to the IMF.

As a direct consequence of Eq. (9), the time-like component of the momentum transfer can be ignored, i.e. one can approximate $q^2 \approx -\vec{q}^2 = -\vec{q}_\perp^2$ and therefore one can again pull $f_{\zeta=0}(x, -\vec{q}_\perp^2)$ out of the integral in Eq. (5), and one finds

$$f_\Psi(x, \vec{q}_\perp) = f_{\zeta=0}(x, -\vec{q}_\perp^2) \int \frac{d^2 p_\perp \Psi^*(\vec{p}_\perp')\Psi(\vec{p}_\perp)}{\sqrt{2E_{\vec{p}} 2 E_{\vec{p}'}}}. \tag{10}$$

Finally making use of the fact that we chose a wave packet that is very localized, i.e. $\Lambda_{\vec{p}_\perp} \gg \Lambda_{\vec{q}_\perp}$, we can approximate $\Psi(\vec{p}_\perp') \approx \Psi(\vec{p}_\perp)$ in Eq. (10) yielding

$$f_\Psi(x, \vec{q}_\perp) = f_{\zeta=0}(x, -\vec{q}_\perp^2). \tag{11}$$

As in the NR case, the dependence on the wave packet has disappeared and the identification has become unique.

We have thus accomplished to show that, OFPDs at $\zeta = 0$ have in the IMF the simple physical interpretation as Fourier transforms of impact parameter dependent parton distributions with respect to the impact parameter. In other words, OFPDs, in the limit of $\zeta \to 0$, allow to simultaneously determine the longitudinal momentum fraction and transverse impact parameter of partons in the target hadron in the IMF.

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[1] Note that, by choice of the wave packet, $\vec{q} \equiv \vec{p}' - \vec{p}$ is always transverse i.e. $q_z = 0$.

3 The transverse center of momentum

In the previous chapter, when we introduced the notion of PDFs as a function of the impact parameter, we did not specify with respect to which point (or line) the impact parameter is defined. In NR scattering experiments, the impact parameter usually refers to the center of mass, but it is perhaps not a priori obvious how to generalize the concept of a $\perp$ center of mass to the IMF.

In the infinite momentum or light-front (LF) frame there exists a residual Galilei invariance under the purely kinematic $\perp$ boosts

$$
\begin{aligned}
x_i &\longrightarrow x'_i = x_i \\
\vec{k}_{i\perp} &\longrightarrow \vec{k}'_{i\perp} \equiv \vec{k}_{i\perp} + x_i \Delta \vec{P}_\perp,
\end{aligned}
\tag{12}
$$

where we denote the longitudinal momentum fraction and $\perp$ momentum of the i-th parton in a given Fock component by x_i and $\vec{k}_{i\perp}$ respectively, the LF-Hamiltonian transforms covariantly, i.e. $P^- - \frac{\vec{P}_\perp^2}{2P^+}$ remains constant, which resembles very much NR boosts

$$
\vec{k}_i \longrightarrow \vec{k}'_i = \vec{k}_i + m_i \Delta \vec{v} = \vec{k}_i + \frac{m_i}{M} \Delta \vec{P},
\tag{13}
$$

with $E - \frac{\vec{P}^2}{2M}$ remaining constant. Because of this similarity between NR boosts and $\perp$ boosts in the IMF, many familiar results about NR boosts can be immediately transferred to the IMF.

For this work, the most important result concerns the physical interpretation of the form factor as the Fourier transform of the charge distribution in a frame where the center of mass

$$
\vec{R}_{CM} \equiv \sum_i \frac{m_i}{M} \vec{r}_i
\tag{14}
$$

is at the origin. One can also rephrase this result by stating that the Fourier transform of the form factor is the distribution of the charge as a function of the distance from $\vec{R}_{CM}$.

By comparing Eqs. (12) and (13) it becomes clear that in the IMF the momentum fractions x_i play the role of the mass fraction $\frac{m_i}{M}$. Hence it is not very surprising that the IMF analog to the NR center of mass $\vec{R}_{CM}$ is given by the *transverse center of momentum*

$$
\vec{R}_\perp \equiv \sum_i x_i \vec{r}_{i\perp},
\tag{15}
$$

i.e. a weighted average of $\perp$ positions, but where the weight factors are given by the (light-cone) momentum fractions and not the mass fractions [2].

The Fourier transform of the $\zeta = 0$ (i.e. 'unskewed') OFPD

$$F(x, \vec{b}_\perp) \equiv \int d^2 q_\perp e^{-i\vec{b}_\perp \cdot \vec{q}_\perp} f_{\zeta=0}(x, -\vec{q}^2) \tag{16}$$

can thus be interpreted as the (light-cone) momentum distribution of partons as a function of the $\perp$ separation from the $\perp$ center of energy-momentum $\vec{R}_\perp = \sum_i x_i \vec{r}_{i\perp}$ of the target. [3]

The above observation about the $\perp$ center of momentum has one immediate consequence for the $x \to 1$ behavior of $F(x, \vec{b}_\perp)$. Since the weight factors in the definition of $\vec{R}_\perp$ are the momentum fractions, any parton i that carries a large fraction x_i of the target's momentum will necessarily have a $\perp$ position $\vec{r}_{i\perp}$ that is close to $\vec{R}_\perp$. Therefore the transverse profile (i.e. the dependence on $\vec{b}_\perp$) of $F(x, \vec{b}_\perp)$ will necessarily become more narrow as $x \to 1$, i.e. we expect that partons become very localized in $\perp$ position as $x \to 1$. By Fourier transform, this also implies that the slope of $f_{\zeta=0}(x, t)$ w.r.t. t at $t = 0$, i.e.

$$\langle \vec{b}_\perp^2 \rangle \equiv 4 \frac{\frac{d}{dt} f_{\zeta=0}(x, t)|_{t=0}}{f_{\zeta=0}(x, 0)} \tag{17}$$

should in fact vanish for $x \to 1$!

4   Explicit example (model calculation)

In order to illustrate the results from the previous sections in a concrete example, we make a Gaussian ansatz for the $\vec{k}_{i\perp}$ dependence of the light-cone wave function in each Fock component [5]

$$\Psi(x_i, \vec{k}_\perp) \propto \exp\left[-a^2 \sum_{i=1}^N \frac{\left(\vec{k}_{i\perp} - x_i \vec{P}_\perp\right)^2}{x_i}\right] \Psi(x_i), \tag{18}$$

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[2] Note that for NR systems the momentum fractions are given by the mass fractions.
[3] The above results can be verified by starting with $\zeta = 0$ OFPDs calculated from a LF Fock space expansion for the hadron state [5].

where $\Psi(x_i)$ is not further specified. The above model ansatz allows to carry out the $\vec{k}_\perp$ momentum integrals explicitly, yielding

$$f(x,t) = \exp\left[\frac{a^2}{2}\frac{1-x}{x}t\right]f(x),$$

(19)

i.e.

$$F(x,\vec{b}_\perp) \propto \frac{x}{1-x}\exp\left[-\frac{1}{2a^2}\frac{x}{1-x}\vec{b}_\perp^2\right]f(x).$$

(20)

Of course, the above model (18) clearly oversimplifies the actual complexity of the nucleon's Fock space wavefunction, but the qualitative behavior of the final result (20) seems reasonable: as predicted in the previous section, the width of $F(x,\vec{b}_\perp)$ in $\perp$ position space goes to zero as $x \to 1$. For decreasing x, the distribution widens monotonically until it diverges for $x \to 0$. Of course, the divergence of the $\perp$ width as $x \to 0$ is probably an artifact of the model ansatz which seems is more suitable for intermediate to large values of x. Nevertheless, it seems reasonable that the $\perp$ width increases with decreasing x since for example the pion cloud, which is expected to contribute only at smaller values of x, should me more spread out than the valence partons, which are expected to dominate at larger values of x.

Notice also that both the above model calculation as well as the pion cloud picture suggest that the $\vec{b}_\perp$-dependence in $F(x,\vec{b}_\perp)$ does not factorize, which of course also translates back into a lack of factorization of the t dependence in $f(x,t)$. A similar lack of factorization was also observed in Ref. [6].

5   Practical aspects

From the experimental point of view, $\zeta = 0$ is not directly accessible in DVCS since one needs some longitudinal momentum transfer in order to convert a virtual photon into a real photon. However, there seem to be several ways around this difficulty. First of all, one can perform DVCS experiments at finite (small) ζ and extrapolate to $\zeta = 0$. Secondly, $\zeta = 0$ can be accessed in real wide angle Compton scattering [7]. Finally, the moments of $F(x,b)$ can be also obtained from information at $\zeta > 0$. First we use that

$$p^{+2}\int_{-1}^{1} dx\, x f_\zeta(x,t) = \left\langle p'\left|\bar{\psi}iD^+\gamma^+\psi\right|p\right\rangle.$$

(21)

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The matrix element of the local operator on the r.h.s. of Eq.(21) can be expressed in terms of invariant form factors. For example, for a spinless target one finds

$$\left\langle p' \left| \bar{\psi} i D^\mu \gamma^\nu \psi \right| p \right\rangle = \bar{p}^\mu \bar{p}^\nu A(t) + \bar{\Delta}^\mu \bar{\Delta}^\nu B(t) + \text{traces}, \tag{22}$$

where $\bar{p} = p + p'$ and $\Delta = p' - p$, i.e. the explicit ζ-dependence has disappeared! One can now determine $A(t)$ and $B(t)$ from the OFPDs at nonzero values of ζ. Once one has determined these invariant form factors, one can go back and evaluate the r.h.s. of Eq. (22), yielding

$$\int_{-1}^{1} dx \, x f_{\zeta=0}(x,t) = A(t). \tag{23}$$

For the slightly more complicated case of a spin $\frac{1}{2}$ target we consider the 2nd moment of the spin-independent OFPD $H(x, \xi, t)$ discussed in Ref. [3]. For the exact definition we refer the reader to Ref. [3]. Note that we switch here briefly to the definitions used in Ref. [3]: Although similar, ξ and ζ differ, but the difference are not very important for this paper as $\xi = 0$ corresponds to $\zeta = 0$. Furthermore, for $\xi \neq 0$, the definition of x is different from ours, but again this does not matter here since the difference vanishes as well for $\zeta = 0$.

As discussed in Ref. [3], the 2nd moment of H can also be expressed in terms of invariant form factors

$$\int_{-1}^{1} dx \, x H(x, \xi, t) = A(t) + \xi^2 C(t). \tag{24}$$

By measuring $\int dx \, x H(x, \xi, t)$ for two different values of ξ and the same value of t, one can determine $A(t)$ and $C(t)$ independently. The quantity of interest for extracting the 2nd moment of the impact parameter dependent distribution is then obtained by evaluating Eq. (24) for $\xi = 0$

$$\int_{-1}^{1} dx \, x H(x, 0, t) = A(t). \tag{25}$$

The generalization of this method to higher moments is straightforward, but more and more form factors need to be introduced. However, the main lesson is that it should be possible to extract the lowest moments $\int dx \, x^n f_{\zeta=0}(x,t)$ *can* be extracted from OFPDs with $\zeta \neq 0$ which then allows one to extract also the moments of the impact parameter dependent distributions $\int dx \, x^n F(x, b)$.

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6 Summary and outlook

Off-forward parton distributions for $\zeta \to 0$, i.e. where the off-forwardness is only in the $\perp$ direction, can be identified with the Fourier transform of impact parameter dependent parton distributions w.r.t. the impact parameter b, i.e. the $\perp$ distance from the center of (longitudinal) momentum in the IMF. This identification is very much analogous to the identification of the charge form factor with the Fourier transform of a charge distribution in position space.

The $\zeta \to 0$ limit of OFPDs is difficult to access in DVCS. However, as we illustrated for the 2nd moment in Sec.5, one can also measure x-moments for nonzero values of ζ and use those to construct moments for $\zeta = 0$.

Knowing the impact parameter dependence allows one to gain information on the spatial distribution of partons inside hadrons and to obtain new insights about the nonperturbative intrinsic structure of hadrons. For example, the pion cloud of the nucleon is expected to contribute more for large values of b. Shadowing of small x parton distributions, is probably stronger at small values of b since partons in the geometric center of the nucleon are more effectively shielded by the surrounding partons that partons far away from the center. These and many other models and intuitive pictures for the parton structure of hadrons give rise to predictions for the impact parameter dependence of PDFs that reflect the underlying microscopic dynamics of these models. Through the experimental measurement of OFPDs using DVCS, it will for the first time become possible to obtain experimental information on the impact parameter dependence of parton structure and which will thus provide much more comprehensive tests on our understanding of nonperturbative parton structure.

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